**Photon Field Quantization**

The photon field Hamiltonian is, in natural units (with c = ℏ = ε0 = 1 – what I’m calling Natural Lorentz Heaviside units I think – see Units file), given by:



where E(x) and B(x) are of course operators now, one for every point in space (and | | means vector magnitude). In order to quantize the field, it is best to start with the action/Lagrangian as we’ve done before. So, recall/note that in these same units we have:



Everything can be written in terms of the space-time vector potential: Aα = (φ, **A**). And we have:



A problem we encounter is that we cannot impose canonical commutation relations on φ, because is missing. The underlying problem is that not all the Aα’s are independent. But we can fix the issue by choosing a gauge to eliminate the extraneous degrees of freedom, at least to the level necessary so that we can successfully quantize the theory.

**Coulomb Gauge**

One option is to go to the Coulomb gauge. We’ll recall from EM that this entails setting ∇·**A** = 0. But now it is also the case that in a source free region, which we do find ourselves in, it also follows that φ = 0 (can plainly see this from the discussion in EM file). We will need both of these conditions to formulate a self-consistent theory. The latter condition is easy to implement. This takes our L to:



But the former isn’t easy to implement in real space. We can do it in Fourier space though. Let’s make the transform,



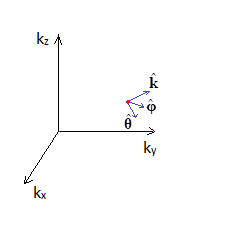
Then the constraint transforms to:



We can expand **A**(**k**,t) in a **k**-dependent basis using a resolution of identity:



which looks like this pictorally,



Note that these directions are the usual basis in spherical coordinates. And can be broken into Cartesian components (in k-space). So let **k** be given by



Then we have:



But it’s more typical to use a slightly different transverse basis.



where **ε**(**k**,λ) are just specified to be orthonormal unit vectors perpendicular to , and also switch signs when **k** → -**k**, just like they did in the phonon case:



Well, **θ** and **φ** do kind of do this themselves. Then we have:



This must be true for all x, and so must be true for all Ak. So we’ll set Ak = 0. Then,



Since **A** is Hermitian, we must have:



And so we must have:



If we plug this into L, we have:



Performing the x integral makes **k**´ = -**k**. And we get:



Now use the cross product property of the unit vectors, to write:



Probably better to rewrite it as:



Now let’s get the equation of motion for our Fourier space operators. Taking the functional derivative of the action



and setting to zero,



[this might be a little suspect as A and A† are correlated, but it gives the right result] Taking the dagger of this, we have a slightly nicer version:



The solution is:



And taking the inverse Fourier transform, we have:



Better to write this as:



We will recognize the coefficient of eiωt as the creation operator, and of e-iωt the annihilation operator, of excitations. And the excitations are indeed ωk = k (which would be ℏkc = pc if we restored the units). Defining,



where we presume,



we can write:



(actually N depends on k, and should go inside the ∫d3k, woops) We’d like to work out these creation/annihilation operators and so we impose canonical commutation relations now. Noting,



These would (have) be(en):



(note this i,j subscript is now referring to x,y,z components) and now plugging in our FFE we have:



Let N = n√ωk, where n is yet to be determined (again, n is k-dependent and should be inside integral, but whatever). Then we have:



and



So now we have:



So we have finally,



When we plug this back into H, I’ll take on faith that we’ll get (see the next file for derivation in other units):



**Lorentz Gauge**

When we come to the problem of incorporating relativity into the mix, we’ll recall that ME were already relativistically invariant – even if not obviously so. So too, this quantization procedure is also relativistically invariant and quite correct in itself. But usually we want to write it in a manifestly relativistically invariant way. So we would quantize using the Lorentz gauge rather than the Coulomb gauge.



We’ll go through this procedure a little differently than above. We’ll implement this gauge as a constraint on the action. So we’d write [keeping in mind the natural units we’re working in]:



where ξ is a sort of Lagrange multiplier. We’d attempt to impose canonical commutation relations. First let’s simplify. Since it appears within the action, we can write:



and so have:



Okay, so now the momentum conjugate to Aα is, keeping in mind ∂0 = ∂0,



And we’d want to impose:



which is the same as:



But we’ll note that we cannot because it is incompatible with the gauge, because operating ∂μ on both sides, we should get 0 on the LHS because of the gauge condition, but we don’t get 0 on the right hand side. We’ll revisit this issue; turns out we can demand two to-be-encountered operators be identical and that will resolve our problem. So for now, let’s work out the equations of motion.



Now we get the nicest equation when we set ξ = 1, so that’s the point of adding the ‘constraint’ term. Then our equation of motion is just:



Now the solution to this equation is, simply borrowing from our work on bosons (specializing to massless case)



where the creation/annihilation operators obey commutation relations:



consistent with the ones with same index position above. But this is not how we will keep it. Basically, we want to use a more convenient (for implementing the gauge condition) orthonormal basis for A, just as we did for the Coulomb gauge. The gauge condition will read:



So we’ll switch to a set of k-dependent (and covariant we’ll say) ortho’normal’ basis vectors which obey the same dot product relation that the (covariant) α do, namely (I’m moving the parameter λ down into the subscript/superscript because now there is an important distinction to be had between covariant and contravariant basis vectors and so we need notation to match):



It follows we have a resolution of identity (recall some properties of covariant/contravariant tensors from tensor file)



(the last line follows only because ηλλ´ is identical to ηλλ´ and further because its diagonal so ηλλ´ = ηλλδλλ´)

I’ll throw in an important identity we’ll use later for good measure.



Here’s another one:



OK we’ll let **0**(**k**) point in the time-like direction, , meaning, that there is *some* reference frame in which = 0.



Then we’ll have **3**(**k**) point in the - plane, perpendicular to of course. We can work this out. First we’ll say:



Then imposing the orthogonality condition we have:



Then the normality condition requires:



And so,



Note that in the frame where = 0, we’d have **3**(**k**) = . Then the other two guys are unspecified beyond the fact that they’re perpendicular to the previous two:



Now we want to put



in that basis. So we can insert a resolution of identity:



Then define the new creation/annihilation operators:



and we can say:



Note the new creation/annihilation operators obey the following commutation relations:



Note the diagonal η’s cancel out because it’s only when λ = λ´ (according to the last η) that is non-zero, and so they equate to ηλληλ´λ´ = (ηλλ)2 = 1. So they’re the same as the old relations. Now we need to impose the gauge condition, in some fashion. So since we cannot impose it identically, we’ll do with a lesser demand, that:



A slightly stronger prescription is:



And this would imply that the other guy in (**x**,t) is zero too. Now let’s consider the dot product. We have:



And so our filling these into our condition,



and so we can impose our gauge condition by making an operator identity:



So this condition reduces our degrees of freedom to 3 technically, but it looks like observables will come out in terms of just two because of this cancelation from the gauge condition. For instance, if we go through the motions of putting forming H and putting the FFE into it, we will get:



(because η00 = -η33)